Propagation of relativistically intense laser pulses in nonuniform plasmas

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(Received 10 October 1997)

We consider the interaction of relativistically intense short laser pulses in a cold-electron–ion plasma. Starting with the fully relativistic equations, we derive, for pancakelike-shaped pulses, a one-dimensional nonlinear Schrödinger equation for arbitrary pulse amplitudes. We show that relativistic effects change the pulse shape along its propagation. In the weakly relativistic case, these pulses can be adequately described by soliton-type solutions. Moreover, plasma inhomogeneities lead to pulse acceleration, which creates radiation fields similar to those of an accelerated charged particle. Two types of waves are emitted by these accelerated laser pulses: low-frequency electromagnetic waves and electron plasma wake fields. Different kinds of mechanisms leading to radiation are identified. [S1063-651X(98)00909-X]

PACS number(s): 52.40.Nk, 52.35.Mw

I. INTRODUCTION

The successes obtained in recent times by high power laser technology, pointing to Peta-Watt lasers in the near future, have opened fields for experimental research in the range of laser field energies comparable to, or exceeding, the electron rest mass energy. The electromagnetic radiation emitted by cosmic objects (such as galactic nuclei, radio galaxies, or quasars) may serve as a source for similar strong field conditions in astrophysics. When such strong radiation acts on a plasma, the electron mass becomes dependent on the amplitude of the pumping waves, leading to considerable changes in the dynamical plasma behavior.

The interaction of relativistically intense electromagnetic waves with a plasma has been a subject of considerable interest [1–8]. The relativistic motion of the electrons leads to the formation of a different type of solitons [9], relativistic modulational and filamentational instabilities [10–13], self-focusing [6,14], and harmonic generation [15]. The relativistic effects occurring in self-focusing can also lead to plasma compression in the region of the moving focus [16]. Stochastic acceleration of electrons moving in strong wave fields can also occur [17,18].

Another interesting aspect of strong nonlinear laser-beam–plasma interactions is the possibility of generating large amplitude plasma waves (the so-called wake fields) [19–25]. These large electrostatic fields can then interact with the nearly resonant electrons and accelerate not only the electrons, but also the photons [26].

However, in spite of the fact that a great deal of work has been devoted to the relativistic effects in plasmas, we are still unable to give answers to many fundamental questions, in particular, to the self-consistent stationary solutions for strong relativistic waves in the two- and three-dimensional cases, even in the weakly relativistic limit. Another outstanding problem is the definition of the group velocity of the laser pulses [27]. We still do not know how to describe the evolution of relativistically intense short pulses in an inhomogeneous plasma. Yet another difficult problem is the definition of the shape of the laser pulse after an interaction. Also, the radiation of low-frequency electromagnetic waves by laser beams is not well understood.

In nonrelativistic plasmas, some of the main features of soliton acceleration are well known [28,29]. In particular, a soliton moving with acceleration can, like a charged particle, emit ion acoustic waves [31]. The generation of Alfvén waves by Langmuir solitons was also considered [30]. The related problem of emission of low-frequency electromagnetic waves by a short laser pulse in a stratified rarefied plasma was also considered in the nonrelativistic approximation [31,32].

In the present paper we shall consider the interaction of a relativistically intense short laser pulse with a cold-electron–ion inhomogeneous plasma, assuming that the ions are at rest. In three dimensions, the pulse shape can significantly change along propagation and is very difficult to describe. We will restrict our analysis to pancake-type pulses (pulses whose transverse length is much longer than longitudinal length) because it was shown [8] that for such a situation the shape distortion in the direction perpendicular to the pulse propagation can be neglected. Taking this into account, we could derive a one-dimensional Schrödinger equation for an arbitrary laser pulse amplitude. Using this equation and assuming a slightly relativistic nonlinearity, we obtained accelerated soliton solutions. Moreover, such accelerated soliton laser pulses can excite electron plasma waves as well as low-frequency electromagnetic waves.

The content of this paper is the following. In Sec. II, starting from Maxwell's equations and the electron equations of motion, we derive two sets of equations for the fast motion, associated with the laser field oscillations, and for the slow motion, due to ponderomotive force effects. In Sec. III we then derive the set of equations adequate for the description of the propagation of pancakelike pulses. In Sec. IV we take the weakly relativistic limit and derive the corresponding accelerated soliton solutions. In Sec. V we discuss the various possible mechanisms for electron plasma wave excitation and for transverse electromagnetic radiation by the ac-

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celerated soliton pulse. Finally, a brief summary of our results is given in Sec. VI.

II. BASIC EQUATIONS

Let us study the interaction of circularly polarized electromagnetic waves with the plasma electrons (the ion motion is neglected) by using Maxwell's equations complemented by the electron fluid equations

$$\vec{\nabla}\times\vec{B}=\frac{1}{c}\frac{\partial\vec{E}}{\partial t}+\frac{4\pi}{c}\vec{j}, \tag{1}$$

$$\vec{\nabla}\times\vec{E}=-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}, \tag{2}$$

$$\vec{j}=-en\vec{v}n=-\frac{en\vec{p}}{m\gamma}, \tag{3}$$

$$\vec{\nabla}\cdot\vec{E}=4\pi e(n_0-n), \tag{4}$$

$$\frac{\partial n}{\partial t}+\vec{\nabla}\cdot(n\vec{v})=0, \tag{5}$$

$$\left[\frac{\partial}{\partial t}+\vec{v}\cdot\nabla\right]\vec{p}=-e\left[\vec{E}+\frac{1}{mc\gamma}\vec{p}\times\vec{B}\right], \tag{6}$$

where $\gamma=\sqrt{1+p^2/m^2c^2}$, $\vec{p}=m\vec{v}/\sqrt{1-v^2/c^2}$, m is the electron rest mass, and the electron temperature is zero. Instead of using Eq. (6) we describe the motion of an electron in an electromagnetic field by the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t}+H-e\phi=0, \tag{7}$$

where S is the action, H is the Hamiltonian function for a particle in an electromagnetic field

$$H=c\sqrt{\left(\vec{P}+\frac{e}{c}\vec{A}\right)^2+m^2c^2}, \tag{8}$$

and $\vec{P}=\vec{\nabla}S=\vec{p}-(e/c)\vec{A}$ is the generalized electron momentum. Taking the gradient of Eq. (7), we obtain the equation of motion of the plasma electrons in the form

$$\frac{\partial\vec{p}}{\partial t}=-e\vec{E}-mc^2\vec{\nabla}\gamma. \tag{9}$$

Equation (9), which is equivalent to Eq. (6), is more appropriate for our calculations. Therefore, from now on our discussion will be based on the set of equations (1)–(5) and (9).

We will assume that the plasma electrons are at rest before the arrival of the strong electromagnetic pulse, which means that we can use the initial condition $\vec{p}(t_0,\vec{r})=0$. Using Eq. (9) we can rewrite Eqs. (1) and (2) in terms of the momentum $\vec{p}$:

$$\frac{\partial}{\partial t}\vec{\nabla}\times\vec{p}=\frac{e}{c}\frac{\partial\vec{B}}{\partial t}, \tag{10}$$

$$-e\frac{\partial\vec{E}}{\partial t}=\frac{\partial^2\vec{p}}{\partial t^2}+mc^2\frac{\partial}{\partial t}\vec{\nabla}\gamma. \tag{11}$$

Obviously, Eq. (10) allows us to write $\vec{\nabla}\times\vec{p}=(e/c)\vec{B}$. From Eqs. (1), (2), and (9)–(11) we obtain the following equation for the electron momentum perturbation associated with the electromagnetic wave pulse:

$$\nabla^2\vec{p}_t-\frac{1}{c^2}\frac{\partial^2\vec{p}}{\partial t^2}-m\frac{\partial}{\partial t}\vec{\nabla}\gamma=\frac{4\pi e^2}{mc^2\gamma}n\vec{p}, \tag{12}$$

where $\vec{p}=\vec{p}_t+\vec{p}_l$ and subscripts l and t denote the longitudinal and the vortex parts of $\vec{p}$.

Let us now use dimensionless variables, defined by the following changes of notation: $p/mc\to p$, $e\phi/mc^2\to\phi$, $n/n_0\to n$, $\vec{r}\omega_p/c\to\vec{r}$, and $\omega_p t\to t$, where $\omega_p=(4\pi e^2 n_0/m)^{1/2}$ is the electron plasma frequency. Equation (12) can now be written as

$$\nabla^2\vec{p}_t-\frac{\partial^2\vec{p}}{\partial t^2}-\frac{\partial}{\partial t}\vec{\nabla}\gamma=\frac{n}{\gamma}\vec{p}, \tag{13}$$

with $\gamma=(1+p^2)^{1/2}$.

When the frequency of the laser pulse ω_0 is much larger than the electron plasma frequency ω_p, we can use the multiple time-scale method [9]. We write every physical variable as $a=\langle a\rangle+\tilde{a}$, where the angular brackets denote an average over a period $\tau=2\pi/\omega_0$. We will assume that these averaged values vary over much longer times, typically $\tau_1=2\pi/\omega_p$. Under the condition

$$\tilde{\gamma}=\frac{\vec{p}_t\cdot\langle\vec{p}\rangle}{\langle\gamma\rangle}\ll\langle\gamma\rangle,$$

where $\langle\gamma\rangle=(1+p_t^2+\langle\vec{p}\rangle^2)^{1/2}$ [14,15], we obtain for the time averaged variables the evolution equations

$$\nabla^2\langle\vec{p}_t\rangle-\frac{\partial^2}{\partial t^2}\langle\vec{p}\rangle-\frac{\partial}{\partial t}\vec{\nabla}\langle\gamma\rangle=\frac{\langle n\rangle}{\langle\gamma\rangle}\langle\vec{p}\rangle, \tag{14}$$

$$\frac{\partial\langle n\rangle}{\partial t}+\vec{\nabla}\cdot\frac{\langle n\rangle}{\langle\gamma\rangle}\langle\vec{p}\rangle=0, \tag{15}$$

$$\frac{\partial}{\partial t}\langle\vec{p}\rangle=-\langle\vec{E}\rangle-\vec{\nabla}\langle\gamma\rangle, \tag{16}$$

$$\nabla^2\phi=\langle n\rangle-1. \tag{17}$$

The equations for the rapidly varying quantities are obtained from Eq. (13):

$$\nabla^2\vec{p}_t-\frac{\partial^2\vec{p}_t}{\partial t^2}=\frac{\langle n\rangle}{\langle\gamma\rangle}\vec{p}_t, \tag{18}$$

$$\vec{B}=\vec{\nabla}\times\vec{p}_t, \tag{19}$$

$$\frac{\partial \vec{p}_t}{\partial t} = -\vec{E}, \tag{20}$$

where we have used the dimensionless variables defined by $e\vec{B}/mc\omega_p \rightarrow \vec{B}$ and $e\vec{E}/mc\omega_p \rightarrow \vec{E}$. The fast transverse momentum can be represented by

$$\vec{p}_t = \frac{1}{\sqrt{2}}(\hat{e}_x + i\hat{e}_y)p(r_\perp, z, t)e^{i(k_0 z - \omega_0 t)} + \text{c.c.} \tag{21}$$

and the amplitude $p(r_\perp,z,t)$ can be written as $p = ae^{i\psi}$, where the quantities $a(r_\perp,z,t)$ and $\psi(r_\perp,z,t)$ are real. From Eq. (16) we can also write for the longitudinal part of the slow motion

$$\frac{\partial}{\partial t}\langle p_\perp^l \rangle = \vec{\nabla}_\perp(\phi - \langle \gamma \rangle), \tag{22}$$

$$\frac{\partial}{\partial t}\langle p_z^l \rangle = \vec{\nabla}_z(\phi - \langle \gamma \rangle). \tag{23}$$

Let us now multiply Eq. (22) by $\vec{\nabla}_z$ and Eq. (23) by $\vec{\nabla}_\perp$. Summing the resulting equations, we can easily get [8]

$$\nabla_z \langle p_\perp^l \rangle = \vec{\nabla}_\perp \langle p_z^l \rangle. \tag{24}$$

This relation allows us to compare $\langle p_\perp^l \rangle$ and $\langle p_z^l \rangle$. If the averaged physical quantities change faster along the direction of propagation of the laser pulse than in the perpendicular direction (which is the case for pancakelike-shaped pulses) it follows from Eq. (24) that $\langle p_\perp^l \rangle \ll \langle p_z^l \rangle$:

$$\frac{\langle p_\perp^l \rangle}{\langle p_z^l \rangle} \sim \frac{\Delta z}{\Delta r_\perp} \sim \frac{z - v_g t}{r_\perp} \ll 1. \tag{25}$$

Such a condition is satisfied in a variety of different physical situations related to very short laser pulses, such as self-focusing, self-channeling, or the generation of wake fields. We will further assume that $\hat{p}_\perp^2 \gg (p_z^l)^2$, which means that we can write $\langle \gamma \rangle = (1 + a^2)^{1/2}$. Condition (25) also corresponds to assuming that the laser pulses change mainly along the direction of its propagation and not in the perpendicular directions. Assuming that Eq. (25) stays valid, we can then write

$$p_t = \frac{1}{\sqrt{2}} p_\perp(r_\perp) p_\parallel(z,t)(\hat{e}_x + i\hat{e}_y)e^{i(k_0 z - \omega_0 t)}, \tag{26}$$

where the perpendicular part of the vortex momentum perturbation can be written as $p_\perp(r_\perp = \sqrt{I_0}e^{-r_\perp^2/2r_0^2})$, for an initial value of the laser intensity defined by a Gaussian profile, such that

$$p_0^2 = a_0^2 = I_0 e^{-r_\perp^2/r_0^2 - \xi^2/l_0^2}, \tag{27}$$

where $\xi = z - [k_z c^2/\omega(k)]t$, $r_\perp = \sqrt{x^2 + y^2}$, r_0 and l_0 are the transverse and longitudinal pulse dimensions, and $p_\parallel(z,t)$ is an arbitrary function describing the longitudinal pulse profile, to be determined later. The electron plasma density can be written as

$$\langle n \rangle = n_0 + \Delta n(z) + \delta n(r_\perp, z, t), \tag{28}$$

where n_0 is a constant, $\Delta n(z)$ characterizes the inhomogeneity profile, and $\delta n(r_\perp, z, t) = n_\perp(r_\perp) n_\parallel(z,t)$ is the perturbation associated with the ponderomotive force effects.

III. DERIVATION OF PROPAGATION EQUATIONS FOR PANCAKE PULSES

So far, we have derived the general equations governing pulse propagation in the relativistic regime. We now focus our attention over pancakelike pulses. In this case, condition (25) is valid and several different analytic results can be derived.

Substituting p_t from Eq. (26) in Eq. (18), multiplying the resulting equation by $p_\perp 2\pi r_\perp dr_\perp$, and integrating over $r_\perp$, we obtain

$$2i\omega_0\left(\frac{\partial}{\partial t} + \frac{k_0}{\omega_0}\frac{\partial}{\partial z}\right)p_\parallel + \left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2}\right)p_\parallel = [\beta(p_\parallel)n_\parallel - 1]p_\parallel, \tag{29}$$

where we have used

$$\beta(p_\parallel)n_\parallel = [1 + \Delta n(z)]\frac{1}{B}\int_0^\infty \frac{p_\perp^2}{\langle \gamma \rangle} 2\pi r_\perp dr_\perp + \frac{n_\parallel(z,t)}{B}\int_0^\infty \frac{p_\perp^2 n_\perp(r_\perp)}{\langle \gamma \rangle} 2\pi r_\perp dr_\perp, \tag{30}$$

$$B = \int p_\perp^2 2\pi r_\perp dr_\perp \tag{31}$$

and $k_0/\omega_0 = v_g$ is the ''group'' velocity along the axis Oz. For future use, we will also define

$$k_\perp^2 = \frac{1}{B}\int \left(\frac{\partial p_\perp}{\partial r_\perp}\right)^2 2\pi r_\perp dr_\perp. \tag{32}$$

Introducing new variables $(\xi = z - v_g t, t)$ instead of (z,t), Eq. (29) becomes

$$2i\omega_0 \frac{\partial}{\partial t}p_\parallel + \frac{1}{\gamma_g^2}\frac{\partial^2}{\partial \xi^2}p_\parallel + 2v_g \frac{\partial^2 p_\parallel}{\partial t \partial \xi} = (\beta n_\parallel - 1)p_\parallel, \tag{33}$$

where we have $\gamma_g = (1 - v_g^2)^{-1/2}$. If we now use $p_\parallel = a_\parallel e^{i\phi}$, then it follows from Eq. (33) that

$$\frac{\partial}{\partial t}\left\{\left(\omega_0 + v_g\frac{\partial \phi}{\partial \xi}\right)a_\parallel^2\right\} + \frac{\partial}{\partial \xi}\left\{\left(\frac{1}{\gamma_g^2}\frac{\partial \phi}{\partial \xi} + v_g\frac{\partial \phi}{\partial t}\right)a_\parallel^2\right\} = 0. \tag{34}$$

From here follows the conservation law

$$\int d\xi\left(\omega_0 + v_g\frac{\partial \phi}{\partial \xi}\right)a_\parallel^2 = \text{const.} \tag{35}$$

From Eq. (33) we can also derive an equation for the phase ϕ, which takes the form

$$-2\omega_0 \frac{\partial \phi}{\partial t} a_\parallel + \frac{1}{\gamma_g^2}\left[\frac{\partial^2}{\partial \xi^2} a_\parallel - \left(\frac{\partial \phi}{\partial \xi}\right)^2 a_\parallel\right] + 2v_g\left(\frac{\partial^2 a_\parallel}{\partial t \partial \xi} - \frac{\partial \phi}{\partial t}\frac{\partial \phi}{\partial \xi} a_\parallel\right) = [\beta n_\parallel - 1] a_\parallel . \tag{36}$$

Equations (34) and (36) can be reduced to much simpler expressions if we notice that we have $\omega_0 \gg |v_g(\partial/\partial\xi)|$, $|\partial\phi/\partial\xi| \gg |\partial\phi/\partial t|$, and $|\partial a_\parallel/\partial\xi| \gg |\partial a_\parallel/\partial t|$, which are valid for underdense plasma and in the quasistationary approximation. Introducing a new variable $x = \gamma_g \xi$, we finally have

$$\omega_0 \frac{\partial a_\parallel^2}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x} a_\parallel^2\right) = 0 \tag{37}$$

and

$$-2\omega_0 \frac{\partial \phi}{\partial t} a_\parallel + \frac{\partial^2 a_\parallel}{\partial x^2} - \left(\frac{\partial \phi}{\partial x}\right)^2 a_\parallel = [\beta(a_\parallel) n_\parallel - 1] a_\parallel . \tag{38}$$

The solutions for these two equations can be taken in the form $a_\parallel[x - \overline{x(t)}]$ and $\phi(x,t)$, where $\overline{x(t)}$ is the coordinate of the center of the laser pulse. The time evolution of this coordinate will be determined in the following way. Since the amplitude is assumed to be a function dependent only on the self-similar argument $\eta = x - \overline{x(t)}$ and retaining only solutions that vanish at infinity ($\eta \to \infty, a_\parallel = 0$), we conclude from Eq. (37) that

$$\phi(x,t) = \omega_0 \dot{\overline{x}}(t) x + F(t), \tag{39}$$

where $F(t)$ is a function of time that can be considered arbitrary, for the time being, but will be specified later. Substituting Eq. (39) into Eq. (36), we obtain a closed equation for the amplitude $a_\parallel$:

$$\frac{\partial^2 a_\parallel}{\partial x^2} - \left\{2\omega_0 \dot{F}(t) + \omega_0^2 \dot{\overline{x}}^2 + 2\omega_0^2 x \frac{d^2 \overline{x(t)}}{dt^2}\right\} a_\parallel = [\beta n_\parallel - 1] a_\parallel . \tag{40}$$

Thus, starting from the fully relativistic three-dimensional equations, we could derive, for pancakelike-shaped pulses, a one-dimensional nonlinear Schrödinger-type equation for arbitrary pulse amplitudes.

IV. ACCELERATED SOLITON SOLUTIONS

We now consider the above equations in the weakly relativistic limit, when the analytical solutions for the problem can be carried out until the end. In this case, we can write

$$\langle \gamma \rangle \simeq 1 - \frac{a^2}{2} = 1 - \frac{1}{2} a_\perp^2(r_\perp) a_\parallel^2(x - \overline{x}) \tag{41}$$

and from Eqs. (15)–(17), we can write for δn

$$\left(\frac{\partial^2}{\partial t^2} + 1\right)\delta n = \nabla^2 \frac{a^2}{2}. \tag{42}$$

In this equation we can isolate that part of the density perturbation that is concentrated in the pulse region δn_p from the density perturbation associated with electron plasma waves left behind the pulse (usually called the wake field) N:

$$\delta n = \delta n_p + N. \tag{43}$$

Equation (42) then leads to

$$\delta n_p = \nabla^2 \frac{a^2}{2} \equiv \frac{1}{2}\left(\frac{1}{r_\perp}\frac{\partial}{\partial r_\perp} r_\perp \frac{\partial}{\partial r_\perp} + \gamma_g^2 \frac{\partial^2}{\partial x^2}\right) \times a_\perp^2(r_\perp) a_\parallel^2[x - \overline{x(t)}], \tag{44}$$

$$\left(\frac{\partial^2}{\partial t^2} + 1\right) N = -\frac{1}{2}\frac{d^2 \overline{x}}{dt^2} \nabla^2 \frac{\partial}{\partial x} a^2. \tag{45}$$

In the case of a pancakelike pulse, we can simplify Eq. (44) to

$$\delta n_p \simeq \frac{1}{2}\gamma_g^2 a_\perp^2(r_\perp) \frac{\partial^2}{\partial x^2} a_\parallel^2 \tag{46}$$

and, from Eq. (44), we can write

$$\delta n_p = n_\perp(r_\perp) n_\parallel[x - \overline{x(t)}], \tag{47}$$

where

$$n_\perp(r_\perp) = a_\perp^2(r_\perp). \tag{48}$$

A solution of Eq. (45) that satisfies the condition of the absence of a wake field at $t \to -\infty$ takes the form

$$N(\vec{r},t) = -\frac{1}{2} a_\perp^2(r_\perp) \frac{\partial^3}{\partial x^3} \int_{-\infty}^{t} dt' \, dt'' \frac{d^2 \overline{x}}{dt^2} \sin(t - t') a_\parallel^2[x - \overline{x(t')}]. \tag{49}$$

Now we can return to Eq. (40) and consider the case of a weakly relativistic correction of the electron motion in the electromagnetic field (26). In this case, for $a_\perp = a_{0\perp} \exp(-r_\perp^2/2r_0^2)$, we have

$$\beta n_\parallel - 1 = \Delta n(x) - \frac{1}{2} a_{0\perp}^2 \left(\frac{1}{2} a_\parallel^2 - n_\parallel\right). \tag{50}$$

Then Eq. (40) becomes

$$\frac{\partial^2 a_\parallel}{\partial x^2} - \left\{2\omega_0 \dot{F}(t) + \omega_0^2 \dot{\overline{x}}^2 + 2\omega_0^2 x \frac{d^2 x(t)}{dt^2}\right\} a_\parallel - \left\{\Delta n(x) + \frac{a_{0\perp}^2}{2}\left(n_\parallel - \frac{a_\parallel^2}{2}\right)\right\} a_\parallel = 0, \tag{51}$$

where we have used $n_\parallel = (\gamma_g^2/2)(\partial^2 a_\parallel^2/\partial x^2)$. For the first integral of Eq. (51) we find

$$(1-\gamma_g^2 a_{0\perp}^2 a_\parallel^2)\left(\frac{\partial a_\parallel}{\partial x}\right)^2 - \left\{2\omega_0 \dot{F}(t) + \omega_0^2 \overline{x^2} + 2\omega_0^2 x \frac{d^2\overline{x}}{dt^2}\right\} a_\parallel^2 + \frac{1}{8} a_{0\perp}^2 a_\parallel^2 - \Delta n(x) a_\parallel^2 + \int_{-\infty}^{x} dx' \left(2\omega_0^2 \frac{d^2\overline{x}}{dt^2} + \frac{\partial \Delta n(x')}{\partial x'}\right) a_\parallel^2 = 0. \quad (52)$$

It follows from Eq. (52) that

$$\int_{-\infty}^{\infty} dx \left(2\omega_0^2 \frac{d^2\overline{x}}{dt^2} + \frac{\partial \Delta n(x)}{\partial x}\right) a_\parallel^2 = 0. \quad (53)$$

If we assume that the maximum value of the amplitude $a_\parallel = a_m$ corresponds to the point $x=\overline{x(t)}$, we can obtain from Eq. (52) an equation for the function $\dot{F}(t)$:

$$2\omega_0 \dot{F}(t) + \omega_0^2 \overline{x^2} + 2\omega_0^2 \overline{x} \frac{d^2\overline{x}}{dt^2} + \Delta n(\overline{x}) = \frac{1}{8} a_{0\perp}^2 a_m^2 + \frac{1}{a_m^2} \int_{-\infty}^{\overline{x}} dx \left(2\omega_0^2 \frac{d^2\overline{x}}{dt^2} + \frac{\partial \Delta n(\overline{x})}{\partial x}\right) a_\parallel^2(x). \quad (54)$$

Replacing this equation in Eq. (51), we obtain

$$\frac{\partial^2 a_\parallel}{\partial x^2} - \left\{\frac{1}{8} a_{0\perp}^2 a_m^2 + 2\omega_0^2 [x - \overline{x(t)}] \frac{d^2\overline{x(t)}}{dt^2} + \Delta n(x) - \Delta n(\overline{x}) + \frac{1}{a_m^2} \int_{-\infty}^{\overline{x}} dx \left(2\omega_0^2 \frac{d^2\overline{x(t)}}{dt^2} + \frac{\partial \Delta n(x)}{\partial x}\right) a_\parallel^2 \right\} a_\parallel - \frac{a_{0\perp}^2}{4}\left(\gamma_g^2 \frac{\partial^2 a_\parallel^2}{\partial x^2} - a_\parallel^2\right) a_\parallel = 0. \quad (55)$$

We shall assume in the following that the density of the inhomogeneous plasma changes over distances of the order of the width of the pulse. We can then use the expansion

$$\Delta n(x) = \Delta n(\overline{x}) + (x - \overline{x}) \frac{\partial \Delta n(x)}{\partial x}\bigg|_{x=\overline{x}} + \cdots \quad (56)$$

and retain only the first three terms. If the conditions

$$a_m^2 \gg l_0^2 \frac{\partial^2 \Delta n}{\partial x^2}\bigg|_{x=\overline{x}}, \quad \frac{1}{a_m^2}\int_{-\infty}^{\overline{x}} (x-\overline{x}) a_\parallel^2 dx \ll 1 \quad (57)$$

are satisfied, we obtain for the amplitude of $a_\parallel$

$$\frac{\partial^2 a_\parallel}{\partial x^2} - \frac{a_{0\perp}^2 a_m^2}{8} a_\parallel + \frac{a_{0\perp}^2}{4} a_\parallel^3 - \frac{\gamma_g^2 a_{0\perp}^2}{4} \frac{\partial^2 a_\parallel^2}{\partial x^2} a_\parallel = 0. \quad (58)$$

If we neglect the last term in this equation, which describes the variation of electron density due to ponderomotive force terms, the relativistic corrections alone lead to the soliton solution

$$a_\parallel = \frac{a_m}{ch\left(\dfrac{x - \overline{x(t)}}{d}\right)}, \quad (59)$$

where the soliton width is equal to $d = 2\sqrt{2}/a_{0\perp} a_m$ and the coordinate of the soliton center is, according to Eqs. (53) and (56), given by

$$\frac{d^2\overline{x}}{dt^2} = -\frac{1}{2\omega_0^2} \frac{\partial \Delta n}{\partial x}\bigg|_{x=\overline{x}}. \quad (60)$$

This equation shows that a plasma inhomogeneity implies an acceleration of the soliton. Replacing Eqs. (59) and (60) in Eq. (49), we can find the explicit form of the density distribution associated with the emitted electron plasma waves.

We now analyze the soliton motion for specific forms of the density profile of an inhomogeneous plasma. For a linear profile defined by $\Delta n(x) = 2\omega_0^2 \alpha_c x$, the motion of the soliton center is described, according to Eq. (60), by the formula

$$\overline{x(t)} = \overline{x_0} + v_g t - \frac{1}{2}\alpha_c t^2, \quad (61)$$

where $\overline{x_0}$ is the initial value of the $\overline{x}$ coordinate and the soliton acceleration is constant and equal to $-\alpha_c$. This means that, depending on the sign of the electron density gradient along the soliton propagation, the soliton will slow down if $\alpha_c > 0$ and will accelerate when $\alpha_c < 0$. The soliton velocity will then be $v = v_g - \alpha_c t$.

Another interesting case corresponds to a quadratic inhomogeneity $\Delta n(x) = \omega_0^2 \alpha_c^2 x^2$. Here Eq. (60) shows that the soliton can oscillate periodically around a given point $x=0$.

We now write down the solution of Eq. (58) for the asymptotic condition $a_\parallel(+\infty) = \partial a_\parallel/\partial x|_{x\to+\infty} = 0$. The first integral of Eq. (58) takes the form

$$(1-\alpha u^2)\left(\frac{\partial u}{\partial y}\right)^2 - u^2(1-u^2) = 0, \quad (62)$$

where we have used $\alpha = \gamma_g^2 a_{0\perp}^2 a_m^2/2$, $y = a_{0\perp} a_m x/2\sqrt{2}$, and $u = a_\parallel/a_m$. Equation (62) is interesting because it leads to solutions distinct from Eq. (59) when $u^2 \simeq 1/\alpha$. After integration, we get from Eq. (62)

$$-\frac{1}{2}\ln\frac{\sqrt{1-\alpha u^2}+\sqrt{1-u^2}}{\sqrt{1-\alpha u^2}-\sqrt{1-u^2}} + \alpha^{1/2}\ln(\alpha^{1/2}\sqrt{1-u^2}+\sqrt{1-\alpha u^2}) = \pm[y - \overline{y(t)}]. \quad (63)$$

If we take $\alpha=0$, this equation reduces to Eq. (59).

We can now use Eq. (63) to explore the different qualitative features of the soliton solutions. In Fig. 1 we plot the solution of Eq. (63) as a function of time for the accelerated $\alpha_c < 0$ and decelerated case $\alpha_c > 0$. As expected from the analysis of Eq. (61), the soliton accelerates for $\alpha_c < 0$ [Fig.

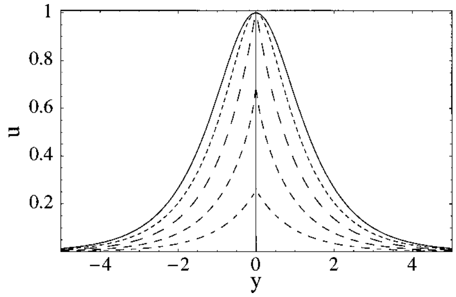

FIG. 1. Soliton solution of Eq. (63) as a function of the parameter α: 0.0 (solid line), 0.5 (short-dashed line), 0.99 (long-dashed line), 2.0 (medium-dashed line), and 3.0 (dash-dotted line).

1(a)], but for $\alpha_c>0$ [Fig. 1(b)] the soliton velocity decreases until the soliton is reflected and then bounces back with its velocity now increasing.

Another important feature is the qualitative dependence of the soliton solutions on the parameter α. For $0<\alpha<1$, the solution is close to the usual solution given by Eq. (59), getting more peaked around the soliton centroid for α close to 1. On the other hand, if $\alpha>1$, cusp soliton solutions are obtained (Fig. 2).

V. RADIATION MECHANISMS

We shall now give a detailed description of the several different mechanisms of emission of low-frequency electromagnetic waves by accelerated short laser pulses in a plasma. We have already mentioned the possibility of excitation of irrotational electric fields and the emission of electron plasma waves in an inhomogeneous plasma, as described by Eq. (49). Now we shall pay attention to the generation of transverse electromagnetic low-frequency waves (or the generation of rotational currents) by ponderomotive forces or by parametric resonance in an inhomogeneous relativistic plasma. We shall show that, even for a homogeneous plasma, relativistic effects due to a short laser pulse can generate low-frequency rotational electromagnetic fields.

In order to write down the equations for the low-frequency magnetic field, we take the rotational electromagnetic field of Eq. (13) and use Eq. (10). The result is

$$\left\{\nabla^2-\frac{\partial^2}{\partial t^2}-\frac{\langle n\rangle}{\langle\gamma\rangle}\right\}\langle\vec{B}\rangle=\vec{\nabla}\left(\frac{\langle n\rangle}{\langle\gamma\rangle}\right)\times\langle\vec{p}\rangle, \tag{64}$$

where

$$\frac{\langle n\rangle}{\langle\gamma\rangle}=\frac{1+\Delta n(x)+\delta n(r_\perp,x,t)}{\sqrt{1+a_\perp^2(r_\perp)a_\parallel^2(z,t)}}. \tag{65}$$

We shall consider Eq. (64) in two different cases. The first one corresponds to nonrelativistic laser pulses for which we can take $\langle\gamma\rangle\simeq1$. We also assume that $|\Delta n(x)|\gg|\delta n|$. Then the emission of low-frequency electromagnetic waves is due to the inhomogeneity of the electron plasma density or the associated acceleration of laser pulses

$$\vec{\nabla}[\Delta n(x)]\times\vec{v}\neq0, \tag{66}$$

where the velocity is determined by Eq. (16).

On the other hand, if $\Delta n(x)=0$ (homogeneous plasma) and if the variation of the density δn due to the ponderomotive force effects is negligible, the mechanism of electromagnetic wave emission is a pure relativistic effect

$$\vec{\nabla}\frac{1}{\langle\gamma\rangle}\times\langle\vec{p}\rangle\neq0. \tag{67}$$

It is then clear that low-frequency electromagnetic waves can be emitted due to either electron plasma density inhomogeneity or relativistic mass correction effects. We also want to mention that the result (67) is valid at a distance from the focus of the laser pulse. In the focal region, $\delta\langle\gamma\rangle\leqslant\delta n$ and the general expression (65) must be used.

Finally, we consider a short laser pulse such that its initial intensity shape can be taken as a rectangular parallelepiped with sides r_0 and e_0. We assume that the electron mass is modulated in time according to

$$m=m_0\langle\gamma\rangle=m_0[1+a_0^2(1+\epsilon\cos2\Omega t)]^{1/2}, \tag{68}$$

where ϵ is the modulation depth and Ω is the modulation frequency of the electromagnetic wave pulse. In this case, we neglect the right-hand side of Eq. (64) and assuming that the electron density modulation is weak when compared to the relativistic factor $\langle\gamma\rangle$, Eq. (64) reduces to

$$\frac{d^2\langle B\rangle}{dt^2}+k^2\langle B\rangle+\frac{\langle B\rangle}{\sqrt{1+a_0^2(1+\epsilon\cos2\Omega t)}}=0. \tag{69}$$

If, furthermore, the modulation depth is small and we have $a_0^2\epsilon\ll1$, this equation takes the form of a Mathieu equation for which well known conditions exist for unstable modes.

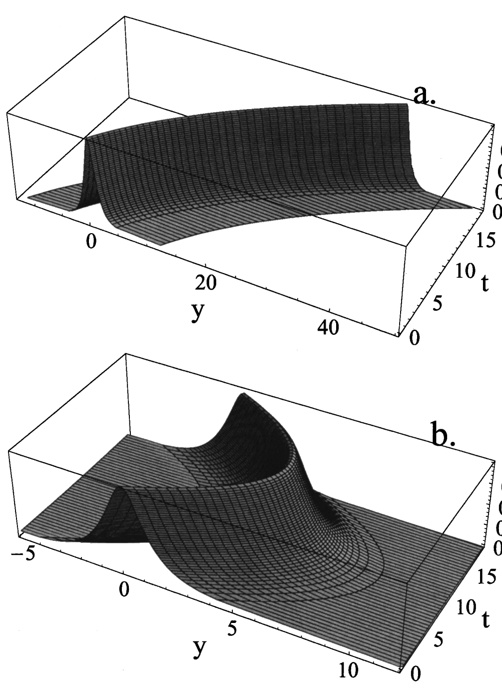

FIG. 2. Soliton evolution as a function of time: (a) accelerated [$\alpha_c=-0.10$ in Eq. (61)] and (b) decelerated ($\alpha_c=+0.10$) motion of the soliton center for $\alpha=0.5$ and $v_g=1$.

A similar equation for the parametric resonance case was derived before [2] and studied numerically [4,33]. It was shown that unstable solutions for Eq. (69) could be found for $\Omega \sim \sqrt{k^2c^2+\omega_p^2} \ll \omega_0$, for the small amplitude case $(a_0^2 \ll 1)$. Ω is the typical frequency of the emitted low-frequency electromagnetic waves. For resonance the growth rate is $\Gamma = \frac{1}{16}(\omega_p^2/\Omega^2)a_0^2$. Time amplitude modulated pulses can then emit low-frequency electromagnetic waves in a more efficient way than the usual mechanisms based on inhomogeneity of the background plasma or ponderomotive force effects.

VI. CONCLUSION

In this work we have considered the interaction of an intense and short laser pulse with an inhomogeneous plasma. We have used a relativistic formulation but considered a cold-electron–ion plasma and neglected the ion motion.

We focused our attention on the important case of pancakelike pulses, such as those currently produced in intense laser-plasma interaction experiments. In this case, a one-dimensional nonlinear Schrödinger equation could be derived for arbitrary pulse amplitudes. Using a purely analytical approach, we were able to show that the pulse shape is changing along the propagation, due to relativistic effects. In the weakly relativistic limit, such pulses could be adequately described by soliton-type solutions.

Moreover, we could show that the plasma inhomogeneities, not only induce the above-mentioned pulse shape variation, but also lead to pulse acceleration, which was previously observed for other kinds of pulses. Such a pulse acceleration produces radiation fields, similar to those of accelerated charged particles. Different kinds of mechanisms of radiation were identified, leading to the emission of low-frequency electromagnetic waves and to the production of electron plasma waves (usually known as wake fields).

Our results can eventually be useful in the interpretation of laser-plasma experiments and in the identification of the observed radiation processes. In particular we suggest that the soliton solutions for the pulse shape and its change along propagation, both derived here, can be identified in experiments and numerical simulations in the near future.

[1] A. I. Akhieser and R. V. Polovin, Sov. Phys. JETP **3**, 696 (1956).

[2] N. L. Tsintsadze, Sov. Phys. JETP **32**, 684 (1971).

[3] C. Max, J. Arons, and A. B. Langdon, Phys. Rev. Lett. **33**, 209 (1974).

[4] A. Bourdier, G. di Bona, N. Fortin, and C. Masselet, Phys. Rev. A **13**, 887 (1976).

[5] J. F. Drake, Y. C. Lee, K. Nishikawa, and N. L. Tsintsadze, Phys. Rev. Lett. **36**, 196 (1976); J. F. Drake, Y. C. Lee, and N. L. Tsintsadze, *ibid.* **36**, 31 (1976).

[6] P. K. Shukla, N. N. Rao, M. Y. Yu, and N. L. Tsintsadze, Phys. Rep. **138**, 1 (1986).

[7] T. Katsouleas and W. Mori, Phys. Rev. Lett. **61**, 90 (1988).

[8] L. Tsintsadze, *Proceedings of the International Topical Conference on Research Trends in Coherent Radiation Generation and Particle Accelerators, La Jolla, 1991*, edited by J. M. Buzzi, P. Sprangle, and K. Wille (American Institute of Physics, New York, 1992), p. 474.

[9] V. I. Berezhiani, N. L. Tsintsadze, and D. D. Tskhakaya, J. Plasma Phys. **24**, 15 (1980).

[10] L. N. Tsintsadze, Sov. J. Plasma Phys. **17**, 872 (1991).

[11] P. K. Shukla, R. Bharuthram, and N. L. Tsintsadze, Phys. Rev. A **35**, 4889 (1987); Phys. Scr. **38**, 578 (1988).

[12] W. B. Mori, C. D. Decker, D. E. Hinkel, and T. Katsouleas, Phys. Rev. Lett. **72**, 1482 (1994).

[13] S. Guerin, P. Mora, J. C. Adam, A. Heron, and G. Laval, Phys. Plasmas **3**, 2693 (1996).

[14] D. P. Garuchava, Z. I. Rostomashvili, and N. L. Tsintsadze, Sov. Fiz. Plasmy **12**, 1482 (1986); D. P. Garuchava, I. G. Murusidze, G. I. Suramlishvili, N. L. Tsintsadze, and D. D. Tskhakaya, J. Plasma Phys. **59**, 57 (1998).

[15] P. Sprangle, E. Esarey, J. Krall, and G. Joyce, Phys. Rev. Lett. **69**, 2200 (1992).

[16] D. P. Garuchava, N. L. Tsintsadze, and D. D. Tskhakaya, Sov. Phys. JETP **71**, 873 (1990).

[17] J. T. Mendonça, Phys. Rev. A **28**, 3592 (1983).

[18] J. T. Mendonça and L. Oliveira e Silva, Phys. Rev. E **55**, 1217 (1997).

[19] T. Tajima and J. M. Dawson, Phys. Rev. Lett. **43**, 267 (1979).

[20] P. Chen, J. M. Dawson, R. W. Huff, and T. Katsouleas, Phys. Rev. Lett. **54**, 693 (1985).

[21] P. Sprangle, E. Esarey, A. Ting, and G. Joyce, Appl. Phys. Lett. **53**, 2746 (1988).

[22] V. I. Berezhiani and I. G. Murusidze, Phys. Lett. A **148**, 338 (1990).

[23] P. Sprangle, E. Esarey, and A. Ting, Phys. Rev. A **41**, 4463 (1990).

[24] S. V. Bulanov, V. I. Kirsanov, and A. C. Sakharov, Sov. J. Plasma Phys. **16**, 935 (1990).

[25] I. G. Murusidze and L. N. Tsintsadze, J. Plasma Phys. **48**, 391 (1992).

[26] L. Oliveira e Silva and J. T. Mendosa, Phys. Rev. E **57**, 3423 (1998); J. M. Dias *et al.*, Phys. Rev. Lett. **78**, 4773 (1997).

[27] C. D. Decker and W. B. Mori, Phys. Rev. Lett. **72**, 490 (1994); Phys. Rev. E **51**, 1364 (1995).

[28] H. H. Chen and C. S. Liu, Phys. Fluids **21**, 377 (1978).

[29] P. K. Kaw, N. L. Tsintsadze, and D. D. Tskhakaya, Sov. Phys. JETP **55**, 839 (1982).

[30] I. G. Murusidze, N. L. Tsintsadze, and D. D. Tskhakaya, Sov. Fiz. Plasmy **11**, 1 (1985).

[31] L. M. Gorbunov and A. A. Frolov, Sov. Phys. JETP **83**, 967 (1996).

[32] J.T. Mendonça, K. Hizanidis, and D. Frantzeskakis, Opt. Commun. **146**, 245 (1998).

[33] M. Dobrowolny, A. Ferrari, and G. Bosia, Plasma Phys. **18**, 441 (1976); S. Guerin, P. Mora, and G. Laval, Phys. Plasmas **5**, 376 (1998).